THE GEODETIC EFFECT ALONG POLAR ORBITS IN THE KERR SPACETIME

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A gyroscope following a closed polar orbit in the Kerr spacetime is considered. An exact expression is derived giving the shift of the gyroscope's orientation per revolution in terms of the mass and angular momentum parameters of the Kerr metric and the orbit's coordinate radius.

As first realized in the context of the Thirring and Lense [1] approximate solution of Einstein's equations, the magnetic-like components of the gravitational field of a rotating body produce a set of characteristic effects which provide, in principle, new ground for testing the theory of general relativity. One such effect consists of the dragging of the line of nodes of bound orbits in the direction of rotation of the center of attraction and bears the name of Lense and Thirring (LT). Another relates to the well-known geodetic effect [2], which refers to the precession of a gyroscope's spin when the gyroscope falls freely in a gravitational field. Specifically, the central body's angular momentum influences the rate and direction of the gyroscope's precession by an amount given by the so-called "Schiff term" in a well established approximate formula for the total rate of precession [3].

Sakina and Chiba [4] have derived an exact expression for the gyroscope precession in the case of an equatorial circular orbit in the Kerr spacetime. In the present paper we consider the geodetic effect in the same geometry but for the case of a spherical polar orbit. The interest in polar timelike orbits derives from two sources. The first consists of the fact that, when the carrier of the gyroscope follows such an orbit, both the geodetic and the LT effect are simultaneously present. The second is their pertinence to gravitational experiments.

Specifically, while nodal dragging and spin precession are in general quite small, recent technological advances seem to have rendered both measurable even in the case of the weak gravitational field of the earth. This has led van Patten and Everitt [5] and Schiff [6] to propose respective experiments for measuring these effects, and the one devised by Schiff is presently conducted by Everitt, Fiarbank and their collaborators [7] at Stanford University.

Since both experiments involve artificial sattelites in polar orbit around the earth we expect the results of our analysis to shed light on qualitative aspects of these experiments such as the way in which the geodetic and LT effects intertwine.

Consider, then, a timelike geodesic $C(\tau)$, τ denoting proper time, which at $\tau=0$ crosses the symmetry axis of the Kerr spacetime. In the (t, r, θ, ϕ) coordinate system of Boyer and Lindquist this axis lies along $\sin\theta=0$ and we will assume that $\theta(0)=0$. We will further assume that $C(\tau)$ is subsequently confined to the $r=$ const hypersurface. This guarantees that after some time, T_τ say, the geodesic will return to the starting spatial point. Carter's first integrals of motion for such a spherical geodesic read [8,9]

$$\dot{t}=AE/\Delta\Sigma, \qquad \dot{r}=0, \tag{1,2}$$

$$\Sigma^2\dot{\theta}^2=K-a^2\cos^2\theta-a^2E^2\sin^2\theta, \qquad \dot{\phi}=2MaEr/\Delta\Sigma, \tag{3,4}$$

where the dot denotes differentiation with respect to proper time,

$$\Sigma = r^2 + a^2 \cos^2\theta, \tag{5}$$

$$\Delta = r^2 - 2Mr + a^2, \tag{6}$$

$$A = (r^2 + a^2)^2 - a^2\Delta \sin^2\theta, \tag{7}$$

and E, K stand for the energy per unit mass at infinity and Carter's constant, respectively. In our case, these constants are determined by the orbit's coordinate radius, r, and the metric's mass and angular momentum parameters, m and a, via the relations [9]

$$E^2 = \Delta(r^2 + K)/(a^2 + r^2)^2 \tag{8}$$

and

$$K = (Mr^4 - a^2r^3 - 3Ma^2r^2 + a^4r)/(r^3 - 3Mr^2 + a^2r + Ma^2). \tag{9}$$

Assume, now, that a gyroscope falls freely along $C(\tau)$. Then its spin vector $\boldsymbol{S}(\tau)$ is parallelly transported along the given geodesic. This, in turn, implies that, in a parallelly transported orthonormal tetrad $\{\boldsymbol{\lambda}_{(a)}\}$, $a = 0, 1, 2, 3$, with $\boldsymbol{\lambda}_{(0)}$ tangent to $C(\tau)$, the gyroscope's spin stays constant and $\boldsymbol{S}^{(0)} = 0$. Relative to an arbitrary comoving frame, on the other hand, the vector $\boldsymbol{S}$ will precess. In order to describe this precession relative to the asymptotic frame of the Kerr spacetime, we will construct a comoving tetrad which is uniquely related to the coordinate axes at each point of the gyroscope's orbit.

We start this construction by considering the base $\{\boldsymbol{e}_a\}$ where

$$\boldsymbol{e}_0 = (A/\Sigma\Delta)^{1/2}\partial_t + \left[2Mar/(A\Sigma\Delta)^{1/2}\right]\partial_\phi, \qquad \boldsymbol{e}_1 = (\Delta/\Sigma)^{1/2}\partial_r, \tag{9,10}$$

$$\boldsymbol{e}_2 = (1/\Sigma)^{1/2}\partial_\theta, \qquad \boldsymbol{e}_3 = (\Sigma/a)^{1/2}(1/\sin\theta)\partial_\phi. \tag{11,12}$$

In this base the Kerr metric takes the form

$$\mathrm{d}s^2 = \eta_{ab}\boldsymbol{e}^a \otimes \boldsymbol{e}^b, \tag{13}$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ and $\{\boldsymbol{e}^a\}$ are one-forms dual to $\{\boldsymbol{e}_a\}$.

Then, according to (1)–(4), the vectors

$$\boldsymbol{e}_{\hat{0}} = P\boldsymbol{e}_0 + Q\boldsymbol{e}_2, \qquad \boldsymbol{e}_{\hat{1}} = \boldsymbol{e}_1, \tag{14,15}$$

$$\boldsymbol{e}_{\hat{2}} = Q\boldsymbol{e}_0 + P\boldsymbol{e}_2, \qquad \boldsymbol{e}_{\hat{3}} = \boldsymbol{e}_3, \tag{16,17}$$

where

$$P = (A/\Sigma\Delta)^{1/2}E \quad \text{and} \quad Q = \Sigma^{1/2}\dot{\theta} \tag{18}$$

form a comoving frame along $C(\tau)$.

This frame, however, is not well defined on the symmetry axis, because the base $\{\boldsymbol{e}_a\}$ is expressed in terms of the Boyer–Lindquist coordinate system and the latter is singular at $\sin\theta = 0$. But, it is well known that this is a coordinate singularity and this can be seen explicitly by writting the metric in the Kerr–Schild coordinates (x^0, x, y, z). On the symmetry axis the metric reads [10]

$$\mathrm{d}s^2 = -\left[1 - 2Mz/(z^2 + a^2)\right] \mathrm{d}(x^0)^2 + \left[1 - 2Mz/(z^2 + a^2\right]^{-1} \mathrm{d}z^2 + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{19}$$

while, as $\sin\theta \to 0$,

$$\boldsymbol{e}_0 \to \left[1 - 2Mz/(z^2 + a^2)\right]^{-1/2}\partial_{x^0}, \qquad \boldsymbol{e}_1 \to \left[1 - 2Mz/(z^2 + a^2)\right]^{1/2}\partial_z, \tag{20,21}$$

$$\boldsymbol{e}_2 \to \cos\phi\ \partial_x + \sin\phi\ \partial_y, \qquad \boldsymbol{e}_3 \to -\sin\phi\ \partial_x + \cos\phi\ \partial_y. \tag{22,23}$$

Thus, by choosing the direction along which the orbit emerges from the z-axis one can smoothly join the orthonormal base $\{e_a\}$ given by (9)–(12) to a unique coordinate-tied tetrad there. We will assume that initially $\phi = 0$. Then, the gyroscope will return to the starting point on the z-axis along a direction which is determined by putting

$$\phi = 8MaErK(k)/\Delta L^{1/2} \tag{24}$$

in (22) and (23), where $K(k)$ denotes the complete elliptic integral of the first kind,

$$k^2 = a^2(1-E^2)/L, \qquad L = K - a^2E^2. \tag{25,26}$$

The value of ϕ given in (24) is obtained by integrating the equations of motion (1)–(4) [9,11,12], and is equal to the angle by which the line of nodes advances each time the gyroscope completes an oscillation in latitude.

We can, now, make use of Marck's [13] construction of a parallelly transported orthonormal tetrad along an arbitrary timelike geodesic of the Kerr spacetime in order to express the vectors $\{\boldsymbol{\lambda}_{(a)}\}$ in the coordinate-tied comoving base $\{e_{\hat{a}}\}$. Obviously, $\boldsymbol{\lambda}_{(0)} = e_{\hat{0}}$, while for the spacelike $\{\boldsymbol{\lambda}_{(i)}\}$, $i = 1, 2, 3$, Marck's solution gives

$$\boldsymbol{\lambda}_{(1)} = \cos\Psi(\tau)\ \boldsymbol{\lambda}'_{(1)} - \sin\Psi(\tau)\ \boldsymbol{\lambda}'_{(3)}, \tag{27}$$

$$\boldsymbol{\lambda}_{(2)} = P(1/KA)^{1/2}(r^2+a^2)a\cos\theta\ e_{\hat{1}} - (\Delta/KA)^{1/2}ar\sin\theta\ e_{\hat{2}} - Q(1/KA)^{1/2}(r^2+a^2)r\ e_{\hat{3}}, \tag{28}$$

$$\boldsymbol{\lambda}_{(3)} = \sin\Psi(\tau)\ \boldsymbol{\lambda}'_{(1)} + \cos\Psi(\tau)\ \boldsymbol{\lambda}'_{(3)}, \tag{29}$$

where the angle $\Psi(\tau)$ is determined by the equation

$$\dot{\Psi} = EK^{1/2}(K-a^2)/(r^2+K)(K-a^2\cos^2\theta), \tag{30}$$

$$\boldsymbol{\lambda}'_{(1)} = \alpha P(1/KA)^{1/2}(r^2+a^2)re_{\hat{1}} + \beta(\Delta/KA)^{1/2}a^2\sin\theta\cos\theta\ e_{\hat{2}}$$
$$+ \beta Q(1/KA)^{1/2}(r^2+a^2)a\cos\theta\ e_{\hat{3}}, \tag{31}$$

$$\boldsymbol{\lambda}'_{(3)} = \beta Q[\Sigma/A(K+r^2)]^{1/2}(r^2+a^2)\ e_{\hat{2}} - \beta[\Sigma\Delta/A(K+r^2)]^{1/2}a\sin\theta\ e_{\hat{3}}, \tag{32}$$

where

$$\alpha^2 = \beta^{-2} = (K - a^2\cos^2\theta)/(K+r^2). \tag{33}$$

Integrating (30) over a complete oscillation in latitude, we obtain

$$\Psi = -\epsilon(\dot{\theta})4E(K-a^2)\{(K+r^2)\Pi(k,n) - KK(k)\}/(KL)^{1/2}(K+r^2), \tag{34}$$

where $\epsilon(\dot{\theta})$ is the sign of $\dot{\theta}$,

$$n = a^2/K, \tag{35}$$

and $\Pi(k, n)$ denotes the complete elliptic integral of the third kind [14].

Let us assume that $\Psi(0) = 0$ and the $S^{\hat{i}}(0)$ are the components of the gyroscope's spin at that instant. Then it follows from (27)–(32) that, upon the gyroscope's return to the starting point, the components of the spin vector in the comoving frame $\{e_{\hat{i}}\}$ will have changed to $S^{\hat{i}}(T_\tau)$, where

$$S^{\hat{i}}(T_\tau) = R^{\hat{i}}_{\hat{j}}S^{\hat{j}}(0), \tag{36}$$

with the matrix $R^{\hat{i}}{}_{\hat{j}}$ being equal to

$$R^{\hat{i}}{}_{\hat{j}} = \begin{pmatrix} 1+(\cos\Psi-1)\cos^2 Z & -\sin\Psi\cos Z & (\cos\Psi-1)\sin Z\cos Z \\ \sin\Psi\cos Z & \cos\Psi & \sin\Psi\sin Z \\ (\cos\Psi-1)\sin Z\cos Z & -\sin\Psi\sin Z & 1+(\cos\Psi-1)\sin^2 Z \end{pmatrix}, \tag{37}$$

where

$$\cot Z = (a/r)(K+r^2)^{1/2}/(K-a^2)^{1/2}. \tag{38}$$

This means that in each revolution of the gyroscope about the gravitating center its spin rotates by an angle Ψ around an axis which is inclined by an angle Z relative to $e_{\hat{1}}$ and lies in the $e_{\hat{1}}$–$e_{\hat{3}}$ plane of the comoving frame $\{e_{\hat{i}}\}$. During the same interval the frame $\{e_{\hat{i}}\}$ itself rotates with respect to the x–y–z coordinate system by an angle ϕ given by (24) about the z-axis which coincides with the vector $e_{\hat{1}}$ at the beginning and the end of the cycle.

Let us note, in conclusion, that the way in which the angular momentum parameter a influences the geodetic effect is made clear by letting a vanish in (24), (34) and (38). Then one obtains the values 0, $\pi/2$ and $-2\pi(1-3M/r)^{1/2}$ for ϕ, Z, and Ψ, respectively, which correspond to a planar orbit with the gyroscope's spin rotating about an axis normal to the orbit's plane. This is the well-known result for the geodetic effect in the Schwarzschild spacetime [2].

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